Dual-frequency vortex-induced vibrations of long flexible stepped cylinders

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ABSTRACT
Flexible structures within a non-uniform inflow may undergo complex vortex-induced vibrations (VIV) containing multiple frequencies and vibration modes. Therefore, the critical question arises on whether and how a flexible cylinder’s structural response and fluid forces undergoing multi-frequency vibrations resemble or differ from mono-frequency vibrations. Therefore, we experimentally studied the problem of dual-frequency VIV of a stepped flexible cylinder, viz., a large-aspect ratio flexible cylinder consisting of two segments with different diameters and rigid cylinder forced vibration experiments. The results show that the maximum in-line (IL) and cross-flow (CF) displacements and the frequency ratio of the stepped cylinder separated by individual frequency resemble those of a uniform cylinder vibrating in the uniform flow at a single frequency. In addition, it is found that forced vibration results from rigid cylinders undergoing multi-frequency IL and CF motion can improve the prediction of the multi-frequency flexible cylinder VIV, provided the amplitudes and phases, as well as the true reduced velocity \( V_r \), for each of the dual frequencies match, especially when \( V_r \in [4, 8] \).

I. INTRODUCTION
Long slender structures within an oncoming cross-flow (CF) are subject to vibrations caused by vortical structures forming due to a distributed flow instability in their wake. Referred to as the flexible cylinder’s vortex-induced vibrations (VIV), the problem has a considerable theoretical interest as it constitutes a fundamental nonlinear flow–structure interaction (FSI) system. At the same time, it is an essential component of the design of offshore industry systems, such as for the free vibration of a flexibly mounted rigid cylinder (initial, upper, and lower regimes) could be observed. At the same time, the IL motion was strongly correlated with the CF amplitude. Subsequently, two separately conducted experiments on a flexible stepped cylinder model with different aspect ratios \( (L/d) \) (where \( L \) and \( d \) are the length and the diameter of the cylinder), showed vibration response at a narrow-band single frequency, with the IL motion affecting the CF vibration significantly. In addition, wake visualization behind oscillating flexible cylinders using particle image velocimetry and high-fidelity numerical simulations displayed wake patterns of “2S,” “2P,” and “P” configurations, similar to the finding in the rigid cylinder free and forced vibrations. Furthermore, IL motion may significantly change the phase between the shedding of vortices and the cylinder CF motion. In order to achieve high-mode vibrations similar to deepwater marine risers in operation, larger-scale experiments are required. Such experiments revealed new phenomena and physics of very long flexible cylinder VIV, including chaotic vibration patterns, traveling wave-dominated response, and strong higher harmonics structural response in the CF direction.

A series of flexible cylinder experiments revealed that when the bending stiffness was significant, similar amplitude response branches for the free vibration of a flexibly mounted rigid cylinder (initial, upper, and lower regimes) could be observed. At the same time, the IL motion was strongly correlated with the CF amplitude. Subsequently, two separately conducted experiments on a flexible stepped cylinder model with different aspect ratios \( (L/d) \) (where \( L \) and \( d \) are the length and the diameter of the cylinder), showed vibration response at a narrow-band single frequency, with the IL motion affecting the CF vibration significantly. In addition, wake visualization behind oscillating flexible cylinders using particle image velocimetry and high-fidelity numerical simulations displayed wake patterns of “2S,” “2P,” and “P” configurations, similar to the finding in the rigid cylinder free and forced vibrations. Furthermore, IL motion may significantly change the phase between the shedding of vortices and the cylinder CF motion. In order to achieve high-mode vibrations similar to deepwater marine risers in operation, larger-scale experiments are required. Such experiments revealed new phenomena and physics of very long flexible cylinder VIV, including chaotic vibration patterns, traveling wave-dominated response, and strong higher harmonics structural response in the CF direction.

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One of the problems that few past research addressed is for non-uniform inflow. In the ocean, currents of uniform velocity profile are never met. On the contrary, highly sheared flow profiles are found,\textsuperscript{22,23} which introduce multiple vortex shedding frequencies along with the flexible cylinder, exciting multi-frequency vibrations. Field observations\textsuperscript{34} found that under sheared flow conditions, the response can be summarized as a broad-handed frequency response with strong span-wise traveling wave patterns, while the average response amplitude is smaller than that in uniform current.\textsuperscript{25} Later several large-scale high-mode laboratory experiments\textsuperscript{18,26} used a rotation rig to create a well-controlled linearly sheared inflow condition. It was found that the flexible cylinder responded with a broad-handed single frequency, instead of coexisting multiple frequencies. A similar response was observed in simulation,\textsuperscript{35} noting that the lock-in region of the flexible cylinder tended to locate in the high-velocity region. Recently,\textsuperscript{36} a study simulated a flexible cylinder in both linearly and non-linearly sheared flows at a low Reynolds number. The results demonstrated a basic difference\textsuperscript{37} between the two sheared inflow conditions: in the linearly sheared flow time-varying instantaneously, mono-frequency response dominates, while in an exponentially sheared flow, multiple frequencies coexist.\textsuperscript{38} The phase $\theta$ between the IL and the CF trajectory\textsuperscript{35} played an important role in affecting the energy transfer between the fluid and the structure.

To predict the aforementioned complex VIV response of flexible cylinders, the state-of-art semi-empirical prediction codes\textsuperscript{2,33} were used to obtain a database of hydrodynamic coefficients obtained from forced vibration experiments on rigid cylinders.\textsuperscript{31} In such experiments, a rigid cylinder is forced to vibrate at prescribed trajectories,\textsuperscript{33} and hydrodynamic coefficients are measured, including the mean drag coefficient $C_d$, the lift/drag coefficient in-phase with velocity $C_{ls}/C_{ds}$, and the added mass coefficient in the CF/IL direction $C_m/C_m$.\textsuperscript{12,34,35} The majority of the rigid cylinder forced vibration to focus on mono-frequency vibration with constant amplitude. Only a few published works investigate the cylinder vibrating at multiple frequencies. Nevertheless, the multi-frequency rigid cylinder forced vibration experiments\textsuperscript{34,35,40} have shown that the structural response, fluid force, and vortex pattern can be significantly altered. In addition, it is necessary to point out that to the best of the authors’ knowledge, there is no current work studying the rigid cylinder vibrating in both the IL and the CF directions at multiple frequencies, letting alone comparing the fluid force between the rigid and flexible cylinder undergoing multi-frequency VIVs.

Therefore, in order to study flexible cylinders undergoing stable multi-frequency vibration, we experimentally investigate two tension-dominated stepped flexible cylinders, viz., consisting of two segments of unequal diameter, placed in uniform flow.\textsuperscript{20} We then compare the structural response between the stepped and uniform flexible cylinders in the uniform flow, as well as the fluid forces distribution along the stepped flexible cylinder with those measured in the rigid cylinder, forced multi-frequency vibration experiments. This paper is organized as follows: Sec. II presents the experimental methods and models. Section III discusses the experimental results with an emphasis on the connection between the rigid cylinder and the flexible cylinder. Section IV summarizes the main findings of this paper. In the Appendices, we document the structural response of the uniform and the stepped cylinder in the uniform flow, and we provide samples of the comparison of the hydrodynamic forces between the rigid cylinder and flexible cylinder at a given location undergoing multi-frequency vibration.

II. MATERIALS AND METHODS

A. Experiment on the stepped flexible cylinder in uniform flow

The first stepped flexible model, which we refer to as "Step-58," is selected to have $d_1 = 0.5$ cm and $d_2 = 0.8$ cm, shown in Fig. 1(a); and the second flexible model, referred to as "Step-54," has $d_1 = 0.5$ cm and $d_2 = 0.4$ cm, as shown in Fig. 1(a). The diameter $d_1 = 0.5$ cm is used as the reference diameter $d$ in the rest of the paper. The length of the two stepped flexible cylinder models is selected to be the same as $L = 122$ cm, and the length of different diameter sections for the two flexible models is kept as half of the entire model length, namely, $L/2 = 61$ cm.

The flexible models are constructed/molded via urethane rubber (density of 1.38 g/cm$^3$) mixed with tungsten (density of 19.3 g/cm$^3$) powder to alter the mass ratios. For model Step-54, the mass ratios of the two sections with different diameters are chosen to be the same along the span as $m_1^* = m_2^* = 3.56$ ($m_1^*$ is the mass ratio for model Step-54 of the $d_1$ section, and $m_2^*$ is the mass ratio of the $d_2$ section). For Step-58, different mass ratios are selected for the two diameter sections that $m_1^* = 4.0$ was selected for $d_1 = 0.5$ cm section and $m_2^* = 1.38$ was selected for the $d_2 = 0.8$ cm section in order to keep a similar value of mass per unit length along the span. A fishing line is embedded in the center during the molding process, providing sufficient axial stiffness while keeping a low bending stiffness. In the current experiment, the effect of the bending stiffness on the model’s natural frequency and modal shape is negligible compared to that of the tensions applied. Note that the structural damping of the urethane rubber material is higher than the traditional metal or ABS plastic used for the flexible cylinder model in VIV experiments. Nonetheless, we measured the damping ratio of the model in the air by pluck test, and the top tension of the model is set to be the same as that in the still water. We found a damping ratio of 8.1% for Step-58 and that 8.3% for Step-54. We set the initial top tension for the model Step-58 and Step-54 as 2.05N and 2.11N, respectively, in the still water. The model parameters of the two stepped flexible are listed in Table I.

![Fig. 1. Experimental setup of the stepped flexible model of two diameters in MIT Tow Tank](a) model Step-58 of $d_1 = 0.5$ cm and $d_2 = 0.8$ cm; (b) model Step-54 of $d_1 = 0.5$ cm and $d_2 = 0.4$ cm.)
TABLE I. Values of the experimental parameters of the stepped flexible model.

<table>
<thead>
<tr>
<th>Model parameters</th>
<th>Step-58</th>
<th>Step-54</th>
</tr>
</thead>
<tbody>
<tr>
<td>First diameter (d = d_1)</td>
<td>0.5 cm</td>
<td>0.5 cm</td>
</tr>
<tr>
<td>Second diameter (d_2)</td>
<td>0.8 cm</td>
<td>0.4 cm</td>
</tr>
<tr>
<td>Length (L)</td>
<td>122 cm</td>
<td>122 cm</td>
</tr>
<tr>
<td>Aspect ratio (L/d)</td>
<td>244</td>
<td>244</td>
</tr>
<tr>
<td>Immersed ratio</td>
<td>98%</td>
<td>98%</td>
</tr>
<tr>
<td>Mass ratio (m_1^*)</td>
<td>4.0</td>
<td>3.56</td>
</tr>
<tr>
<td>Second mass ratio (m_2^*)</td>
<td>1.38</td>
<td>3.56</td>
</tr>
<tr>
<td>Damping ratio (\zeta)</td>
<td>8.1%</td>
<td>8.3%</td>
</tr>
<tr>
<td>Reynolds number (Re)</td>
<td>500–2200</td>
<td>500–2000</td>
</tr>
<tr>
<td>Reduced velocity (U_r)</td>
<td>13.85–34.95</td>
<td>9.16–29.42</td>
</tr>
</tbody>
</table>

We performed experiments at the MIT towing tank facility with a water depth of 1.27 m. An aluminum frame was built to provide mounting points for the bottom of the model, and a six-axis ATI Gamma sensor was installed on top of the frame to measure the model’s top tension in each experimental run. The vertically installed model had a 98% total immersion length, and it was clamped on both ends. For both model Step-58 and model Step-54, a section of \(d_1 = 0.5\) cm is mounted as the top half of the cylinder and \(d_2 = 0.8\) cm or \(d_2 = 0.4\) cm is kept as the lower half in the water. The model was towed to generate a uniform inflow with different speeds, resulting in Reynolds numbers \(Re = Ud/\nu\) from 500 to 2200 for model Step-58 and 500–2000 for model Step-54, where \(U\) is the inflow velocity and \(\nu\) is the fluid kinematic viscosity. The reduced velocity varies from 13.85 to 34.95 for model Step-58 and from 9.16 to 29.42 for model Step-54. Here, the reduced velocity \(U_r\) and the true reduced velocity \(V_r\) are defined as follows:

\[
U_r = \frac{U}{f_{nat}d}, \\
V_r = \frac{U}{f_d d},
\]

where \(f_{nat}\) is the first modal natural frequency, calculated based on the measured mean tension, assuming \(C_m = 1.0\) along the model, and \(f_d\) is the actual vibration frequency measured in the CF direction. Figure 1 shows a sketch of the experimental setup, while Table I lists the experimental parameters.

We applied underwater optical methods using eight high-speed cameras to capture the IL and the CF vibrations (staggered black and white markers) along the model. Compared to strain-gauge and accelerometer measurements in flexible cylinder VIV experiments, the optical tracking system provides both temporally and spatially dense and direct measurements of the model displacement response. Three cameras are installed over 160\(\ell\) downstream of the model to measure the CF vibration, while five cameras are installed 100\(\ell\) beside the model to measure the IL vibration. At the same time, four 1500-lumen underwater lights were installed to provide enough camera background lighting. Corresponding image processing and motion tracking code were developed to capture and follow the trajectory of either white or black markers.

B. Experiment on the rigid cylinder in uniform flow with prescribed motions

We performed rigid cylinder forced vibration experiments at the MIT Sea Grant towing tank facility (towing length: 9 m, tank width: 1 m, and water depth: 1 m). The towing tank is equipped with four motors that allow complicated motion control. One of the motors moves the carriage and can achieve a constant towing speed from 0.02 to 1.5 m/s. The other three motors allow complex linear motion along with perpendicular to the towing direction as well as the rotation motion.

In the current research, a smooth circular cylinder with a diameter \(D\) of 1.5 in. (3.81 cm) and length \(L\) of 18 in. (45.72 cm) is forced with motions in both the IL and the CF directions, while the model is towed at a constant speed of 0.15 m/s, resulting in \(Re = 5715\). Moreover, two types of rigid cylinder forced vibration experiments are conducted.

In the first type of experiment, the prescribed cylinder trajectory is parametric and sinusoidal in both the IL and the CF directions as follows:

\[
Y(t) = A_y \cos(\omega t),
\]

\[
X(t) = A_x \cos(2\omega t + \theta),
\]

in which \(X(t)\) and \(Y(t)\) are the imposed rigid cylinder trajectories in the IL and CF directions, \(A_x\) is the IL amplitude, \(A_y\) is the CF amplitude, \(\omega\) is the CF vibration frequency, while the IL vibration frequency is set to be twice that of the CF vibration, and \(\theta\) is the phase difference between IL and CF motions. A phase \(\theta \in [\pi, 2\pi]\) denotes the cylinder counterclockwise (CCW) trajectory, and \(\theta \in [\pi, 2\pi]\) represents the clockwise (CW) trajectory. Therefore, the five hydrodynamic coefficients of the forced oscillating rigid cylinder (the mean drag coefficient \(C_d\) the lift coefficient in phase with velocity \(C_l\), the drag coefficient in phase with velocity \(C_m\), the added mass coefficient in the IL direction \(C_{m1}\), and the added mass coefficient in the IL direction \(C_{m2}\)) are functions of the four non-dimensional parameters \(A_y/\pi, A_x/\pi, V_r,\) and \(\theta\) and their range in the current experiments are set as \(A_y/\pi \in [0.05, 0.4], A_x/\pi \in [0.05, 1.35], V_r \in [4, 8],\) and \(\theta \in [0, 2\pi]\). The formulation of the hydrodynamic coefficients can be calculated as follows:

\[
C_v = \frac{2}{\pi T_v} \int_{T_v} \left( C(t) \xi(t) \right) dt,
\]

\[
C_m = -\frac{2U^2}{\pi D^2} \int_{T_v} \frac{1}{T_v} \left( \frac{\ddot{C}(t)}{\dot{\psi}(t)} \right) dt,
\]

in which \(C_v\) is the force coefficient in phase with velocity, \(C_m\) is the added mass coefficient, \(T_v\) is the period of the vibration, \(C(t)\) is the fluctuating drag or lift coefficients \([\dddot{C}(t) = \dot{C}(t) - \ddot{C}(t)\), where \(\dot{C}(t)\) is the mean of the drag and lift coefficients], \(\dot{\psi}\) is the IL or the CF motion non-dimensionalized by the cylinder diameter, \(\dot{\psi}\) and \(\ddot{\psi}\) are the first and second derivatives of \(\psi\) with respect to time, namely, the IL or CF non-dimensional velocity and acceleration.

In the second type of experiment, the prescribed cylinder trajectory is based on the measured motion at different locations along the
flexible model span. In order to achieve such a non-sinusoidal complex trajectory, the “position–velocity–time” (PVT) mode is applied to control the motion profile of the motor directly. Both the position and the velocity are specified between a constant time interval \( t_{M} \) predefined, and a linearly changing acceleration profile, namely, a parabolic velocity profile and a cubic position profile, is imposed during the time interval. In order to achieve the similarity between the rigid cylinder forced motion and measured flexible cylinder motion at different locations along the span, the time interval is defined as follows:

\[
t_{M} = \frac{U_{flex} D}{U_{rig} d_{flex} f_{s}},
\]

in which \( U_{flex} \) is the towing velocity of the flexible cylinder, \( U_{rig} \) is the towing velocity of the rigid cylinder, \( d_{flex} \) is the diameter of the flexible model where the motion is measured and imposed for the rigid cylinder, and \( f_{s} \) is the sampling frequency of the flexible cylinder motion measurement (frame rate of the high-speed camera).

Examples of the imposed IL and CF trajectory and the corresponding comparison of the measured fluid forces of the rigid cylinder and inversely constructed fluid force on the flexible cylinder at a given location are provided in Appendix B. Similar to the first type of rigid cylinder experiments, the hydrodynamic coefficients in the second type of rigid cylinder experiments corresponding to the targeted frequency can be defined as follows:

\[
C_{vi} = \frac{2}{T_{v}} \int_{T_{e}} \left( \tilde{C}_i(t) \tilde{\xi}_i(t) \right) dt,
\]

in which \( \tilde{C}_i \) and \( \tilde{\xi}_i \) are the force coefficient in phase with velocity and the added mass coefficient corresponding to the \( i \)th vibration frequency, \( \tilde{C}_i(t) \) is band-filtered fluctuating drag or lift coefficients corresponding to the \( i \)th vibration frequency, and \( \tilde{\xi}_i \) is the band-filtered IL or CF motion non-dimensional motion corresponding to the \( i \)th vibration frequency.

### III. RESULTS AND DISCUSSION

This section presents the results of the frequency and displacement responses of the flexible cylinder undergoing multiple frequency vibrations over a wide range of \( U_r \). Then, using the inverse force reconstruction method, we compare the sectional hydrodynamic coefficient distribution along the flexible model with the hydrodynamic coefficient acquired from (a) rigid cylinder single frequency combined-IL- and CF forced vibration and (b) rigid cylinder forced vibration with a prescribed motion same as the measured trajectories of the flexible cylinder along the span.

#### A. Structural response of the flexible cylinder

We plot the frequency and phase response of the model Step-58 IL and CF displacement at \( U_r = 16.51 \) in Fig. 2. The CF frequency response in Fig. 2(a) shows that the flexible model vibrates at two different narrow-banded frequencies and two vibration modes, the second and the fourth. Additionally, the wavelet synchro-squeezed transform (WSST) is performed on the CF vibrations of the model Step-58 at \( z/d = -160 \) denoted by the black dashed line; (b) the CF phase response along the model span; (c) the IL frequency response along the model span, and the arrow denotes twice of the frequency components found in the CF response in (a); (d) the IL phase response along the model span; (e) the CF displacement over time; and (f) the IL displacement over time.

![Fig. 2. Frequency and phase response along the flexible model Step-58 at \( U_r = 16.51 \): (a) the CF frequency response along the model span with the inset plot of the WSST analysis on the CF motion at \( z/d = -160 \) denoted by the black dashed line; (b) the CF phase response along the model span; (c) the IL frequency response along the model span, and the arrow denotes twice of the frequency components found in the CF response in (a); (d) the IL phase response along the model span; (e) the CF displacement over time; and (f) the IL displacement over time.](image-url)
The result of the WSST is shown in the inset plot of Fig. 2(a) and reveals that at $z/d = -160$, the two frequency components coexist in time. The IL frequency response along the cylinder span is plotted in Fig. 2(c). We find that in the IL direction, apart from frequencies exactly twice those of the CF frequency components [they are pointed out with black arrows in Fig. 2(c)], there are several other frequencies, with one of them being a strong wide-banded low-frequency component.

Furthermore, we calculate the spanwise phase difference between the real and imaginary parts of the FFT analysis at the response frequencies and as plotted in Fig. 2(b) in the CF direction and Fig. 2(d) for the second harmonics in the IL direction. Recall Fig. 1(a) for the experimental setup of the model Step-58: the smaller diameter $d = 0.5$ cm section is located upper half of the setup from $z/d = -122$ to $z/d = 0$, which induces the high-frequency vibration in the CF direction. Therefore, we see that the higher frequency vibration induces a traveling wave response from top to bottom, while the lower frequency vibration (blue) excites a traveling wave response toward the opposite direction from bottom to top, as shown as the solid red line in Fig. 14(b). The red line plots the phase difference of the $f$ vibration component acquired in the FFT analysis and shows that the vibration at the bottom of the model lags from the vibration at the top of the model.

The same analysis has been performed on the model Step-54 at $U_r = 17.88$, and we plot the result in Fig. 3 for both the IL and the CF frequency and phase responses along the model span. Compared to the model Step-58, we find similar phenomena from the model Step-54 experiment with a slight difference in the IL frequency response that the second harmonic terms respond with a broader frequency bandwidth.

The $1/10$ highest peak of the IL and the CF amplitude response along the model span is calculated for both the model Step-58 and the model Step-54 in the current $U_r$ range. In Fig. 4(a), we present a 3D visualization of the total CF displacement as a function of $z/d$ and $U_r$ for the model Step-58. Meanwhile, the amplitude response corresponding to the lower and higher frequency components are separated and plotted in Figs. 4(b) and 4(c). Such results of the IL amplitude response of the model Step-58 and both the IL and the CF amplitude response of the model Step-54 are given in Appendix A for a concise main context. From Fig. 4(a), no vibration node and anti-node can be observed, and hence, no clear vibration mode can be easily identified. This is due to the coexistence of a strong traveling wave response induced by the two different frequency vibrations, corresponding to different vibration mode numbers. After separating the lower and higher frequency responses, the result is plotted in Figs. 4(b) and 4(c). We can see a clear trend of an increasing vibration mode number with an increase in $U_r$. This phenomenon can also be observed for the IL response of the model Step-58 as well as the IL and the CF response of the model Step-54, shown from Figs. 13–15 in Appendix A.

The maximum of the $1/10$ highest peak of the separated IL and the CF displacement response along the span is calculated for both the lower and higher frequency components. The result is plotted against $U_r$ in Fig. 5 together with the ratio of the vibration frequency over the first natural frequency of the flexible model in the still water, assuming $C_m = 1.0$. Cases in the same modal group for the separated IL and the CF displacement response are grouped and labeled with black arrows (we define the cases in the same modal group as those who share the same dominant mode number for both the IL and the CF separated vibration). In addition, the frequency ratios of the modal group “2n:n” are circled out with the black dashed box.

Figures 5(a) and 5(b) plot the CF displacement, and Figs. 5(c) and 5(d) plot the IL displacement of the lower and higher frequency
components for the model Step-58. To begin with, we find that the mode number of the IL and the CF displacement responses increase with $U_r$. Second, after grouping all the cases with the same modal group, we observe the existence of two types of the modal group of $2n:n$, such as "4/2" (the second mode in the CF direction and the fourth mode in the IL direction), and "2n-1:n," such as "5/3" (the third mode in the CF direction and the fifth mode in the IL direction).

Third, in the CF direction of the same modal group, the maximum amplitude along the span increases monotonically. However, when the IL or the CF dominant mode switches, the CF maximum amplitude jumps (usually drops). We note that while the IL mode switch accompanies the CF mode switch, the IL mode jumps twice more frequently to comply with the doubling of the CF mode number.

Meanwhile, we find similar behavior in the IL direction for the higher modal groups (higher $U_r$ cases). Fourth, we observe that the frequency ratio between the vibration frequency and model first natural frequency in the still water remains relatively constant in the same modal group and jumps between two modal groups. In addition, we notice that the frequency ratio of the modal group $2n:n$ (circled out by the black dashed rectangle box) is larger than the model’s predicted nth modal frequency in the still water, assuming $C_m = 1.0$.

The above findings are very similar to those reported in the experiment of a tension dominated uniform flexible cylinder in the constant current. Comparing the maximum amplitude (non-dimensionalized by the true characteristic diameter) between the uniform cylinder and the model Step-58, we find that the maximum of the separated IL or CF displacement is averagely smaller than that of the uniform cylinder in the uniform current. Here, we need to point out that the reference diameter used for non-dimensional amplitude is $d = 0.5$ cm and for the amplitude of the lower frequency component in Figs. 5(a) and 5(c), the true characteristic diameter is $d_2 = 0.8$ cm.

The result of the maximum displacement and frequency ratio for the model Step-54 is plotted in Fig. 5. The findings of the model Step-54 are similar to that of the model Step-58 and the uniform cylinder. However, we note that the maximum amplitude of the separated displacement non-dimensionalized by the true characteristic diameter of the model Step-54 is found to be on average 10%–15% smaller than that of the model Step-58 [for the amplitude of the frequency component $f_2$ in Figs. 5(f) and 5(h), the true characteristic diameter is $d_2 = 0.4$ cm instead of the reference diameter $d = 0.5$ cm].

B. Hydrodynamic coefficients of the flexible cylinder and the forced oscillating rigid cylinder

We first calculate the fluid forces and the oscillating flexible cylinder via the inverse force reconstruction method and then obtain
FIG. 5. The frequency ratio and the maximum amplitude (1/10 highest peak) vs $U_r$ of the separated vibration amplitude corresponding to different frequencies along the span for the flexible model Step-58: (a) frequency component $f_2$ corresponding to the diameter $d_2 = 0.8$ cm in the CF direction; (b) frequency component $f_1$ corresponding to the diameter $d_1 = 0.5$ cm in the CF direction; (c) frequency component $2f_1$ in the IL direction; (d) frequency component $2f_2$ in the IL direction; for the flexible model Step-54 (the third and fourth): (e) frequency component $f_1$ corresponding to the diameter $d_1 = 0.5$ cm in the CF direction; (f) frequency component $f_2$ corresponding to the diameter $d_2 = 0.4$ cm in the CF direction; (g) frequency component $2f_1$ in the IL direction; and (h) frequency component $2f_2$ in the IL direction. Color red, lower frequency component; color blue, higher frequency component; diamonds, CF response; circle, IL response; black stars, the frequency ratio of the vibration frequency either in the IL or the CF direction over the first natural frequency of the cylinder assuming $C_m = 1.0$. Cases in the same "modal group" for the separated CF and IL response (the same dominant mode in both the IL and the CF directions) are labeled together with the black dashed arrow (except for the lower modal groups of 4/2 in the IL direction).
sectional hydrodynamic coefficients. In this subsection, we focus on the fluid force coefficient in the CF direction of the stepped cylinder, especially the lift coefficient in phase with velocity $C_{lv}$, corresponding to each of different frequency components. $C_{lv}$ is the hydrodynamic coefficient that quantifies energy transfer between the fluid and the structure in the CF direction. In detail, when $C_{lv}$ is positive, the fluid transfers the energy to the structure and excites the cylinder CF vibration. On the other hand, when $C_{lv}$ is negative, the fluid acts as a damper to take the energy away from the oscillating structure and, hence, to reduce the vibration amplitude. In addition, we pay special attention to the true reduced velocity $V_r$ and the phase $\theta$ of the IL and CF trajectory along the model span, as it has been shown in both uniform flexible and rigid cylinders in the uniform flow; and $V_r$ and $\theta$ play significant roles in determining the value of $C_{lv}$ when the IL motion is allowed.

$C_{lv}$ distribution along the model Step-58 is calculated from the measured CF total displacement response, as well as the separated (band-filtered) lower and higher frequency components in the CF displacement response and the result over the entire $U_r$ range is plotted in Fig. 6(a) for $C_{lv}$ of the total displacement response, (b) for $C_{lv}$ of the lower frequency displacement response, and (c) for $C_{lv}$ of the higher frequency displacement response. Figure 6(a) shows that positive $C_{lv}$ (contour of $C_{lv} = 0$ is highlighted by a bold dashed line) can be found everywhere along the span over the $U_r$ range. However, after being separated based on the frequency, we find that the positive $C_{lv}$ (energy-in) region for the model Step-58 takes distinctive locations along the cylinder span based on the local cylinder diameters. As shown in Fig. 6(b), lower frequency vibration concentrates at $L/d \in [-244, -122]$, where the large diameter section of $d_2 = 0.8$ cm is, while Fig. 6(c) reveals that the positive $C_{lv}$ region of the higher frequency vibration is found at $L/d \in [-122, 0]$ with a smaller diameter $d_1 = 0.5$ cm. In addition, we observe that in the same modal group for different $U_r$, the distribution pattern of $C_{lv}$ along the cylinder span is similar, for example, $U_r$ from 17 to 22 (modal group 4/2) for the $C_{lv}$ of the lower frequency component, shown in Fig. 6(b).

Similar to the result of the model Step-58, $C_{lv}$ distribution along with the model Step-54 is calculated and plotted in Fig. 7(a) for $C_{lv}$ of the total displacement response, (b) for $C_{lv}$ of the lower frequency displacement response, (c) for $C_{lv}$ of the higher frequency displacement response. However, compared to the result of the model Step-58, separated $C_{lv}$ based on the lower and higher frequency components display a completely different distribution pattern along with the
model Step-54, shown in Figs. 7(b) and 7(c). Unlike the Step-58 model of the distinctively separated positive Clv distribution along the cylinder span corresponding to different diameters, we find that for the model Step-54, the separated positive Clv can be found mixed along the flexible cylinder span at the same location. In other words, the positive Clv of the lower frequency can be found in the smaller diameter region, while the positive Clv of the higher frequency can be found in the larger diameter region as well.

In order to demonstrate different Clv span distribution patterns between the two models, we selected two cases and plotted the Clv distribution of the total and the separated responses in Fig. 8(a) for the Step-58 model of \( U_r = 31.7 \) and (b) for the Step-54 model of \( U_r = 28.1 \). It is clear that the magnitude of the positive and negative Clv for both the total and the separated response along the span of the flexible model Step-58 is larger than that of the model Step-54. Such an observation explains the difference in the separated displacement amplitude between the two flexible models (the maximum of the displacement amplitude along the model span for the model Step-58 is found to be 10%–15% larger than that of the model Step-54, shown in Fig. 5).

We then study a basic difference in the distribution along the cylinder length of the components of Clv at each of the response frequencies, between the Step-58 and the Step-54 models. We calculate the local true reduced velocity \( V_r \) based on the local diameter for the lower and the higher frequencies, as shown in Fig. 8 in red for the lower frequency and in blue for the higher frequency: For \( U_r = 31.7 \) of the model Step-58, at the smaller diameter region of \( L/d \in [-122, 0] \), the two frequency components result in \( V_r = 5.89 \) and \( V_r = 9.18 \), and at the larger diameter region of \( L/d \in [-244, -122] \), they lead to \( V_r = 3.53 \) and \( V_r = 5.52 \), respectively. Figure 8(b) shows that for \( U_r = 28.1 \) of the model Step-54, at the smaller diameter region of \( L/d \in [-244, -122] \), the two frequency components result in \( V_r = 5.49 \) and \( V_r = 6.87 \), and at the bigger diameter region of \( L/d \in [-122, 0] \), the two frequency components lead to \( V_r = 4.59 \) and \( V_r = 5.73 \). The distribution of the Clv components along the span reveals that (a) for the Step-58 model, in each segment only one of the participating frequencies lies within the reduced velocity range of \( V_r \in [4, 8] \), which can provide positive energy transfer, while the other frequency lies outside this range; whereas for the Step-54 model, both frequency components lie within the range \( V_r \in [4, 8] \) at both segments of the model, indicating possible positive energy transfer for both frequency components along the entire span.
We recall that $V_r = \frac{V}{f_d}$, and therefore, the product of the true reduced velocity and the Strouhal number is $V_r S = \left( \frac{V}{f_d} \right) \left( \frac{d}{U} \right) = \frac{d}{U}$, where $f_d$ is the vortex shedding frequency in the wake of a stationary rigid cylinder with the same diameter and in the same velocity $U$. Past experimental and numerical results show that when $V_r$ is close to the value of the inverse of the Strouhal number, there can be positive energy transfer between the structure and the fluid; more specifically, when $V_r \in [4, 8]$ there is positive $C_h$, when the phase $\theta$ between the IL and CF response leads to a CCW trajectory. Therefore, the inherent difference between the model Step-58 and the model Step-54 lies in the fact that only for the latter both frequencies result in reduced velocities within the range $V_r \in [4, 8]$ (potential excitation region).

One of the critical questions of mapping the vortex-induced force on the flexible cylinder is whether the hydrodynamic coefficients acquired from the rigid cylinder force vibration experiment can be used to estimate force distribution along the long flexible cylinder undergoing VIVs.

Therefore, we consider the fluid forces along the stepped cylinder span for two cases, $U_r = 31.7$ for the model Step-58, and $U_r = 28.1$ for the model Step-54, and we compare them to the hydrodynamic coefficients acquired from the two types of rigid cylinder forced vibration. For the first type of rigid cylinder mono-frequency forced vibration, we merely match the rigid cylinder motion with the component of flexible cylinder motion corresponding to the targeted frequency. Four parameters of $\frac{A_T}{A_T}, \theta, \phi$, and $V_r$ are kept the same, while for the flexible cylinder motion, the local diameter is selected as the reference $d$. For the second type experiment of the rigid cylinder forced vibration, measured IL and CF motion at a specific location of the flexible cylinder are directly imposed on the rigid cylinder.

The IL and CF amplitude response, $\theta$, and $C_h$ and $C_{my}$ along the span of the model Step-58 at $U_r = 31.7$ are plotted in Fig. 9 for both the lower frequency component (first row) and the higher frequency component (second row). At $U_r = 31.7$, the lower frequency component of the model Step-58 vibrates at the fourth mode in the CF direction and the seventh mode in the IL direction, and the higher frequency component vibrates at the sixth mode in the CF direction and the 11th mode in the IL direction. We find that the positive $C_h$ regions are distinctly different for the two frequencies, corresponding to the two sections with different diameters. From Fig. 9, by comparing $\theta$ with the $C_h$ distribution, inversely calculated from the flexible cylinder motion corresponding to each frequency, we see that even in the presence of a second frequency component, $\theta$ plays an important role in affecting the $C_h$ value. More specifically, the positive $C_h$ favors $\theta \in [0, \pi]$, namely, CCW trajectory, as shown in the regions of $L/d \in [-180, 140]$ for the lower frequency component and $L/d \in [-100, 0]$ for the higher frequency component. This was also found in simulations of a uniform flexible cylinder placed in non-uniform inflow at low Reynolds number.

We compare $C_{my}$ obtained through inverse force reconstruction from the flexible cylinder motion (solid line) to the type one rigid cylinder forced vibration (dashed line) and the type two rigid cylinder forced vibration (star) for the model Step-58 in Figs. 9(c) and 9(g) for the two frequency components. We find that the sectional $C_{my}$ on the flexible cylinder span matches well with the fluid force acquired from the rigid cylinder experiments. More specifically, for the two-frequency vibration, when only one $V_r$ is in the range of $V_r \in [4, 8]$, the type one experiment of the rigid cylinder mono-frequency vibration can capture the sign and the magnitude of the $C_{my}$ distribution along the flexible model span accurately. Interestingly, a deviation can be observed between the rigid cylinder and flexible cylinder results at $L/d$ around $-180$ for the lower frequency component and $L/d$ around $-60$ for the higher frequency component. The rigid cylinder data predict $C_{my}$ with a largely negative value as $\theta$ displays a CW trajectory, while $C_{my}$ is found for the flexible cylinder to be continuously positive in those regions. We attribute this to the presence of a strong traveling wave response.

In addition, $C_{my}$ is found to undergo significant variations along the span, but the prediction of $C_{my}$ using rigid cylinder results is accurate.

The IL and CF amplitude response, $\theta$, and $C_h$ and $C_{my}$ along the span of the flexible model Step-54 at $U_r = 28.1$ are plotted in Fig. 10 for both the lower frequency component (first row) and the higher frequency component (second row). At $U_r = 28.1$, the lower frequency component of the model Step-54 vibrates in the fifth mode in the CF direction and the ninth mode in the IL direction, and the higher frequency component vibrates in the sixth mode in the CF direction and the 11th mode in the IL direction. Again, by comparing $\theta$ and $C_h$ distribution along the cylinder span in Figs. 10(c) and 10(g) for the lower frequency component (first row) and the higher frequency component (second row), we find that the positive $C_h$ distribution along the flexible cylinder is favorable to $\theta \in [0, \pi]$, namely, CCW trajectory.
trajectory, similar to the result observed in the model Step-58. Such an effect is shown clearly in Fig. 11 that the positive $C_{lv}$ of two models for all experiment cases concentrates in $h^{2/3}$;
p that $/C_{138}$.

Figures 10(c) and 10(g) plot the comparison of $C_{lv}$ obtained using three methods for the model Step-54. We observe that although the type one rigid cylinder can predict well the sign of $C_{lv}$ along the span, as the qualitative influence of $V_r$ and $\theta$ still holds, it overpredicts the magnitude of the positive $C_{lv}$ value, as found in regions of $L_1/d \in [-120, -160]$ for the lower frequency component, and $L_1/d \in [-120, -220]$ for the higher frequency component. Meanwhile, the type two rigid cylinder forced vibration provides better prediction results, capturing both the sign and magnitude of $C_{lv}$ distribution along the span.

To better quantify this result, we define $C_r$ as the ratio between $L_1$ norm of the reconstructed $C_{lv}$ from the flexible cylinder motion and $C_{lv}$ acquired from the two types of rigid cylinder forced vibration at 36 locations from $z/d = -228$ to $z/d = -18$ equally distributed along the span for both the low and the high-frequency components. The equation for $C_r$ is as follows:

$$C_r = \frac{\|C_{lv, \text{flexible}}\|}{\|C_{lv, \text{rigid-1}}\|}$$

FIG. 9. Structural response and the CF hydrodynamic coefficient distribution along the flexible cylinder span of the model Step-58 at $U_r = 31.7$ for the lower frequency vibration component (first row) and higher frequency vibration component (second row): (a) and (e) amplitude response in both the IL (red) and the CF (blue) directions; (b) and (f) phase $\theta$ between the IL and the CF trajectory; (c) and (g) $C_{lv}$ distribution along the span; and (d) and (h) $C_{my}$ distribution along the span. The CF hydrodynamic coefficients along the span are acquired from the solid line, flexible model displacement; dashed line, the type one rigid cylinder forced vibration experiment; and star, the type two rigid cylinder forced vibration experiment.
where $C_{Fr}^{R2}$ is acquired from the type two rigid cylinder forced vibration. The results from nine cases between $Ur = 19.3$ to $Ur = 28.1$ are plotted in Fig. 12 and reveal that $C_r$ for both the low and the high-frequency components is smaller than 1.0 (mostly smaller than 0.5, except for the case of $Ur = 22.2$), showing that the difference between the $C_r$ distribution along the flexible cylinder undergoing two-frequency vibration and the predicted $C_{Fr}$ from type two rigid cylinder forced vibration experiment.
experiment is smaller than the prediction of the type one rigid cylinder experiment.

In summary, we find that when both coexisting VIV frequencies are in the range of \( V_r \in [4, 8] \), the magnitude of the positive \( C_{lv} \) value (the average energy-in from the fluid to the structure over one vibration period) is reduced for both frequency components, compared to mono-frequency VIV vibration at the same motion. This was also observed in the CF-only rigid cylinder forced vibration reported in Refs. 36 and 47. From the limited flow visualization result on the CF-only rigid cylinder vibration, we may infer that the periodically stable vortex shedding behind the oscillating cylinder may be destabilized with the additional frequency. Hence, a single frequency may no longer dominate the wake when both frequencies are in the range of \( V_r \in [4, 8] \). As a result, the average positive energy from the fluid to structure in the CF direction, expressed by \( C_{my} \), is reduced. We conclude that for coexisting two frequency vibration, when both corresponding \( V_r \) are in the range of \( V_r \in [4, 8] \), the type one experiment of the rigid cylinder mono-frequency vibration overpredicts the positive value of \( C_{lv} \). In contrast, the type two experiment improves the prediction as it captures well both the sign and value of \( C_{lv} \) distribution along with the flexible model.

In addition, the result of \( C_{my} \) for the model Step-54 is plotted in Figs. 10(d) and 10(h), and we find that \( C_{my} \) varies less along the span of model Step-54, as compared to that of model Step-58.

IV. CONCLUSION

We studied the multi-frequency vortex-induced vibrations of long flexible cylinders over a wide range of reduced velocities. The flexible cylinder’s non-uniform (stepped) diameter made a stable multi-frequency vibration possible. We addressed some key questions that apply to the prediction of flexible cylinder VIV in non-uniform inflow: what is the structural and fluid force response of a flexible cylinder undergoing multiple instead of single frequency vibrations, and how accurate is the prediction using hydrodynamic data from forced vibrations of rigid cylinders, as follows:

1. When separated by response frequencies, the multi-frequency structural response of the stepped cylinder resembles that of the uniform cylinder in the uniform current undergoing mono-frequency vibration.
2. The true reduced velocity and the phase angle between in-line and cross-flow vibrations play a significant role in affecting the hydrodynamic coefficient when the cylinder undergoing multi-frequency vibration.
3. The database based on mono-frequency forced vibration of rigid cylinders provides accurate estimates of the forces for a flexible cylinder when only one frequency has a positive \( C_{lv} \) contribution. When more than one frequency has positive \( C_{lv} \) contributions, we need to use a new database based on multi-frequency forced vibrations of rigid cylinders.

Overall, the structural response of a stepped flexible cylinder undergoing stable multi-frequency vibrations, decomposed into components for each frequency, resembles that of a uniform cylinder in uniform flow. For each frequency component, with increasing reduced velocity, we find in-line/cross-flow modal groups either in the form
“2n/n” or “2n-1/n,” with the maximum amplitude of the in-line and the cross-flow displacements increasing monotonically, and the frequency ratio between the vibration frequency and the first modal natural frequency keeping a relatively constant value. When there is a switch in the modal group, the maximum amplitudes generally drop, and the frequency ratio jumps. Furthermore, the maximum amplitude for the flexible structure is on average smaller than that of a uniform cylinder in uniform flow, vibrating at a single frequency.

The lift coefficient in phase with velocity $C_{lv}$ for flexible cylinder multi-frequency vibrations, when separated by frequency, is found to be strongly correlated with the phase difference between the in-line and the cross-flow motions along the model span, as also found for the uniform cylinder. The true reduced velocity $V_r$ for each frequency is found to be a key factor to determine the magnitude of the average energy transfer between the fluid and the structure and, therefore, $C_{lv}$. It is important to note that when both vibration frequencies at any location of the stepped flexible cylinder are in the range of $V_r \in [4,8]$, the magnitude of the positive $C_{lv}$ is reduced compared to that of mono-frequency vibration.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Ang Li: Data curation (equal); Investigation (equal); Writing – original draft (equal). Andreas Mentzelopoulos: Investigation (equal); Writing – original draft (equal). Michael S. Triantafyllou: Conceptualization (supporting); Writing – review and editing (supporting). Dixia Fan: Conceptualization (equal); Funding acquisition (equal); Writing – review and editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

APPENDIX A: STRUCTURAL RESPONSE OF THE STEPPED CYLINDER IN THE UNIFORM CURRENT

In Appendix A, first, we give the 3D visualization of the 1/10 highest peak of the CF amplitude response along the span of the flexible cylinder mode Step-54 in Fig. 13 for (a) total displacement response, (b) separated displacement response corresponding to the lower frequency component $f_1$, and (c) separated displacement response corresponding to the higher frequency component $2f_1$.

Second, we present the 3D visualization of the 1/10 highest peak of the IL amplitude response along the span of the flexible cylinder mode Step-58 in Fig. 14 for (a) total displacement response, (b) separated displacement response corresponding to the lower frequency component $f_2$, and (c) separated displacement response corresponding to the higher frequency component $2f_2$.

Finally, we provide the 3D visualization of the 1/10 highest peak of the IL amplitude response along the span of the flexible cylinder mode Step-54 in Fig. 15 for (a) total displacement response, (b) separated displacement response corresponding to the lower frequency component $2f_1$, and (c) separated displacement response corresponding to the higher frequency component $2f_1$. 
For both the IL and the CF displacement response for the two flexible models, due to the coexistence of the two vibration components and the strong traveling wave response, no clear vibration mode can be identified for the total displacement response, but a clear mode can be observed for the separated displacement response corresponding to the lower and higher frequency components.

**APPENDIX B: TYPE II RIGID CYLINDER FORCED VIBRATION**

We performed the experiment of the rigid cylinder forced vibration with a prescribed trajectory based on the measured motion at different locations along the stepped flexible model span.
In this section, we provide examples of the model Step-58 with the imposed rigid cylinder IL and the CF trajectory and a comparison between the measured force coefficient \( (C_d) \) of the rigid cylinder and inversely constructed the sectional force coefficient on the flexible cylinder, shown in Fig. 16.

**REFERENCES**


**FIG. 16.** Rigid cylinder forced vibration with a prescribed trajectory based on the measured move from the flexible cylinder model Step-58 of case \( U_r = 31.7 \) at \( z/d = -84 \) (left column) and at \( z/d = -180 \) (right column): (a) and (b) the IL (red) and the CF (black) motion; (c) and (d) oscillating lift coefficient; and (e) and (f) oscillating drag coefficient. In the plots of fluid force coefficients: color blue, the measured fluid force from the rigid cylinder experiment; and color red, the reconstructed sectional fluid force from the flexible cylinder motion.
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